Generation of Random Variables

MESIO-SIMULATION

Course 2013-14 Term 1

Authors: Jaume Barceló - Lídia Montero

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SAMPLING FROM PROBABILITY DISTRIBUTIONS

Bibliography:

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- **Ch. 8 Sampling from Probability Distributions**
- J. Banks, J.S. Carson and B.L. Nelson, Discrete-Event System Simulation, Prentice-Hall 1999
- **Ch. 9 Random Variate Generation**
- (*) S. M. Ross, Simulation, Academic Press 2002
- **Ch. 5 Generating Continuous Random Variables**
- Handbook of Simulation: Principles, Methodology, Advances, Applications and Practice, Ed. By J. Banks, John Wiley 1998
- **Ch. 5 (by R.C.H. Cheng) Random Variate Generation**

Generating random variates

- Activity of obtaining an observation on (or a realization of) a random variable from desired distribution.
- These distributions are specified as a result of activities discussed in the *Introduction to the Simulation of Systems*.
- Here, we assume that the distributions have been specified; now the question is how to generate random variates with this distributions to run the simulation.
- The basic ingredient needed for *every* method of generating random variates from *any* distribution is a source of IID *U(0,1)* random variates.
- Hence, it is essential that a statistically reliable *U(0,1)* random number generator be available.

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Requirements from a method

Exactness

- As far as possible use methods that results in random variates with exactly the desired distribution.
- Many approximate techniques are available, which should get second priority.
- One may argue that the fitted distributions are approximate anyways, so an approximate generation method should suffice. But still exact methods should be preferred.
- Because of huge computational resources, many exact and efficient algorithms exist.

Requirements from a method

Efficiency

- Efficiency of the algorithm (method) in terms of storage space and execution time.
- Execution time has two components: set-up time and marginal execution time.
- **Set-up time** is the time required to do some initial computing to specify constants or tables that depend on the particular distribution and parameters.
- **Marginal execution time** is the incremental time required to generate each random variate.
- Since in a simulation experiment, we typically generate thousands of random variates, marginal execution time is far more than the set-up time.

Requirements from a method

Complexity

- Of the conceptual as well as implementational factors.
- One must ask whether the potential gain in efficiency that might be experienced by using a more complicated algorithm is worth the extra effort to understand and implement it.
- “Purpose” should be put in context: a more efficient but more complex algorithm might be appropriate for use in permanent software but not for a “one-time” simulation model.

Robustness

- When an algorithm is efficient for all parameter values.

Inverse transformation method

- We wish to generate a random variate X that is continuous and has a distribution function that is continuous and strictly increasing when $0 < F(x) < 1$.
- Let F^{-1} denote the inverse of the function F.
- Then the inverse transformation algorithm is:
  - Generate $U \sim U(0,1)$
  - Return $X = F^{-1}(U)$.
- Let's define the continuous random variable **U=F(x)**
- To show that the returned value X has the desired distribution F, we must the following proposition.
- **PROPOSITION:** ***Let U be a uniform (0,1) random variable. For any*** *continuous distribution function F the random variable X defined by* $X=F^{-1}(U)$ *has distribution F*

Inverse transformation method

- This method can be used when X is discrete too. Here,

$$F(x) = \Pr\{X \leq x\} = \sum_{x_i \leq x} p(x_i).$$

where, $p(x_i) = Pr\{X = x_i\}$ is the probability mass function.

- We assume that X can take only the values $x_1, x_2, \ldots$ such that $x_1 < x_2 < \ldots$
- The algorithm then is:

1. **Generate $U \sim U(0,1)$.**
2. **Determine the smallest integer I such that $U \leq F(x_I)$, and return $X = x_I$.**

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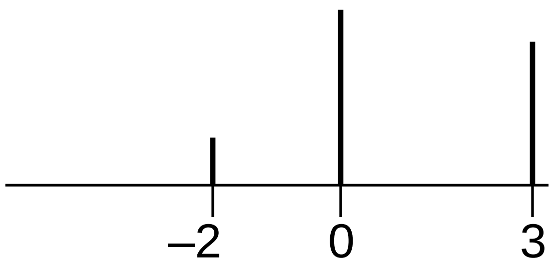
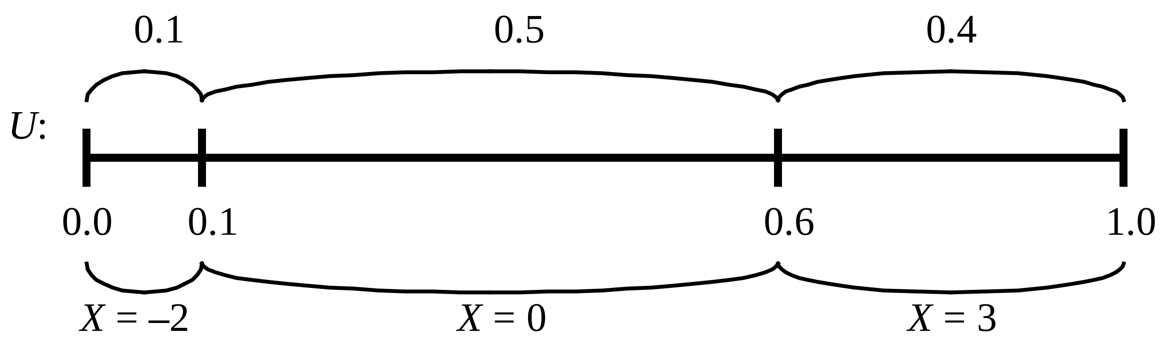
Inverse transformation for Discrete Distributions:

- Example: probability mass function

$$p(x) = P(X = x) = \begin{cases} 0.1 & \text{for } x = -2 \\ 0.5 & \text{for } x = 0 \\ 0.4 & \text{for } x = 3 \end{cases}$$



- Divide [0, 1] into subintervals of length 0.1, 0.5, 0.4; generate U ~ UNIF (0, 1); see which subinterval it's in; return X = corresponding value



Inverse transformation method: Example

- **ALGORITHM:**
  - **Generate U distributed as $\mathcal{U}(0,1)$**
  - **Let X←F^{-1}(U)**
  - **Retunr X**
- Example: exponential distribution

$$f(x) = \lambda e^{-\lambda x}; x \geq 0$$

$$F(x) = \int_0^x f(z)dz = \int_0^x \lambda e^{-\lambda z}dz = -e^{-\lambda z}\Big|_0^x = 1 - e^{-\lambda x}$$

- Let u be U[0,1] then obtain x distributed with pdf f(x) – Exponential solving the following equation:

$$F(x) = u$$

$$u = 1 - e^{-\lambda x} \Rightarrow x = -\frac{1}{\lambda}\ln(1-u) \quad \left(\text{También}: x = -\frac{1}{\lambda}\ln(u)\right)$$

Inverse transformation method: **UNIFORM U [a,b]**

$$f(x) = \frac{1}{b-a}; a \le x \le b$$

$$F(x) = \int_a^x f(z)dz = \int_a^x \frac{dz}{b-a} = \frac{z}{b-a}\Big|_a^x = \frac{x-a}{b-a}$$

- **ALGORITHM:**
  - **Generate U distributed as $\mathcal{U}$(0,1)**
  - **Let X←F^{-1}(U)=a + (b-a) u**
  - **Return X**

$$F(x) = u$$

$$u = \frac{x-a}{b-a} \Rightarrow x = a + (b-a)u$$

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Inverse transformation method: **Weibull(θ,α,β)**

- WEIBULL with LOCATION or SHIFT PARAMETER θ , SHAPE PARAMETER α>0 AND SCALE PARAMETER β>0

$$f(x) = \alpha\beta^{\alpha}x^{\alpha-1}e^{-(\beta(x\text{-}\theta))^{\alpha}}$$

$$F(x) = \int_{\theta}^{x} f(z)dz = 1 - e^{-(\beta(x\text{-}\theta))^{\alpha}} \qquad x \geq \theta$$

- Let u be a random number uniformly distributed in [0,1] . Then to obtain x Weibull (θ, α, β) distributed with pdf f(x) , solve the equation:

$$F(x) = u$$

$$u = 1 - e^{-(\beta(x\text{-}\theta))^{\alpha}} \Rightarrow x = \theta + \left[-\ln(1-u)\right]^{1/\alpha} / \beta$$

- **ALGORITHM:**
  - Generate u=RN(0,1)
  - Return x = θ+[[-ln(1-u)]$^{1/\alpha}$]/β

Inverse transformation method: GEOMETRIC distribution **Geo(p)**

- A discrete distribution, related to Bernoulli process, repetition of i.i.d. Bernoulli experiments, each having success event probability p
- X: Number of trials to obtain a successful event

$$f(x) = p(1-p)^x ; x = 0,1,2,....;0 < p < 1$$

$$F(x) = \sum_{j=0}^{x} p(1-p)^j = p\frac{1-(1-p)^{x+1}}{1-(1-p)} = 1-(1-p)^{x+1}$$

- **Example:** Bernoulli process, tossing a coin and say success event is 'head'. Let us define X Number of trials to get a 'head' X~Geo(p=1/2)
  - Expectation?
  - Variance?

Inverse transformation method: GEOMÉTRIC distribution **Geo(p)**

$$f(x) = p(1-p)^x ; x = 0,1,2,....;0 < p < 1$$

$$F(x) = 1-(1-p)^{x+1}$$

- Let u be a random number uniformly distributed in [0,1] . Then to obtain x integer Geo(p) distributed with pdf f(x) , solve the equation:

$$F(x-1) = 1-(1-p)^x < u \le 1-(1-p)^{x+1} = F(x)$$

$$\Rightarrow \left\{(1-p)^{x+1} \le 1-u < (1-p)^x\right\} \Leftrightarrow \{(x+1)\ln(1-p) \le \ln(1-u) \le x\ln(1-p)\}$$

- Given that 1-p < 1 $\Rightarrow$ ln (1-p) < 0
- **ALGORITHM:**
  - Generate u=RN(0,1)
  - Return x =⌈ [ln(1-u)/ln(1-p)] -1⌉

$$\frac{\ln(1-u)}{\ln(1-p)} - 1 \le x < \frac{\ln(1-u)}{\ln(1-p)}$$

$$\Rightarrow x = \left\lceil \frac{\ln(1-u)}{\ln(1-p)} - 1 \right\rceil$$

RANDOM VARIABLE GENERATION: Inverse transformation approximations

- F(x) cdf complex or analytical expression not existent or we are not able to identify the random variable, so we have a empirical distribution, that gives,
- Approximated F(x) by a look-up table for a set of intervals:

$$(F(x_i),x_i) \text{ s.t. } x_i<x_{i+1}$$

- Combine a search for the interval and a linear interpolation to *F(x) inside the intervals*:
- **ALGORITHM:**

Generate u=RN(0,1)

Find X_i s.t. $F(X_i) \le U \le F(X_{i+1})$

Return X.

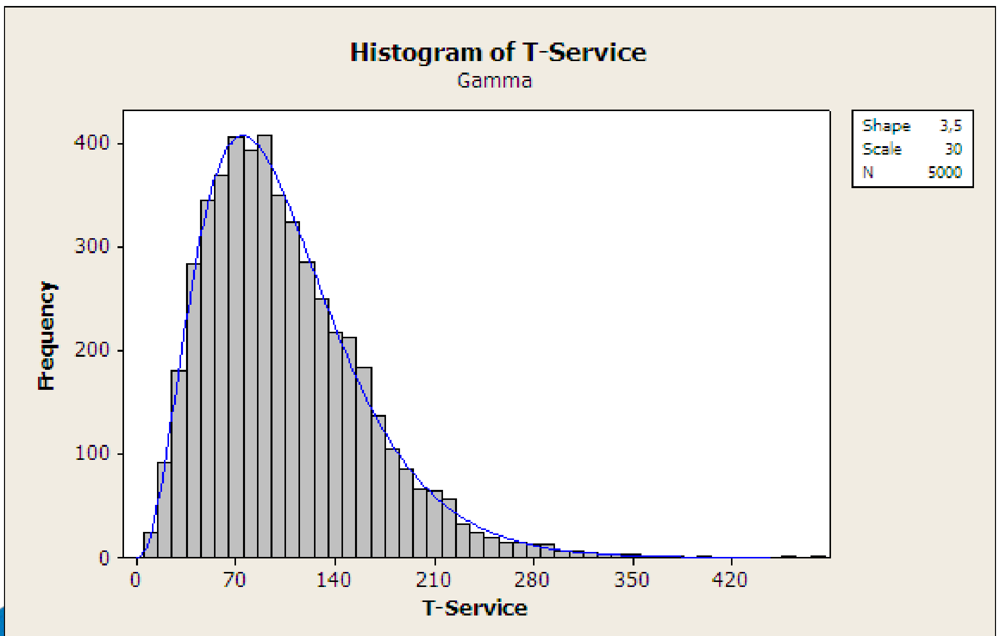
$$X = \frac{[F(X_{i+1}) - U]X_i + [U - F(X_i)]X_{i+1}}{F(X_{i+1}) - F(X_i)}$$

  - Alternative: Solve F(x)-u=0 by a numerical method (Newton-Rawson, Bisection…)

WORKING WITH EMPIRICAL DISTRIBUTIONS (I)

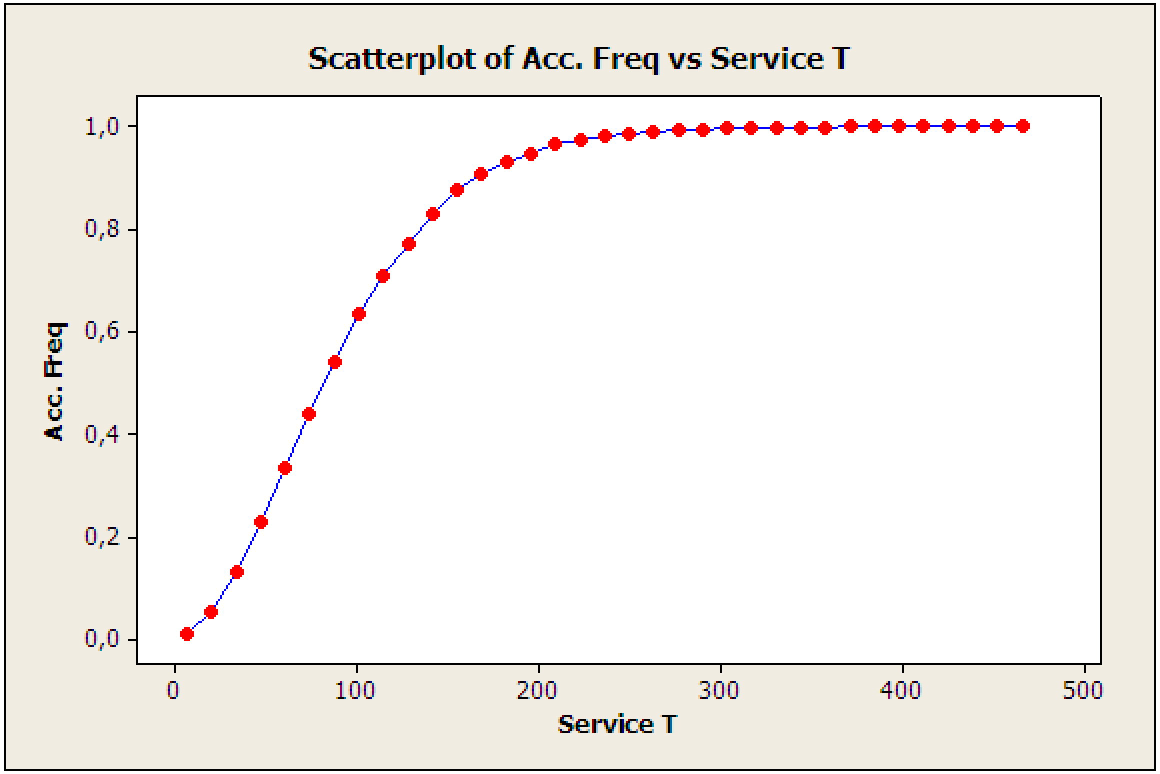
Building the empirical distribution

- **Build a histogram whose ends are, respectively, the smallest and the biggest of the observed values [6.764, 479.356]**
- **Calculate the frequencies and the accumulated frequencies for each class. 35 classes, class width 13.502.**



CLASS	SERVICE TIME	FREQ.	ACC. FREQ
1	6,764 ≤ x < 20,266	0,0114	0,0114
2	20,266 ≤ x < 33,768	0,0422	0,0536
3	33,768 ≤ x < 47,270	0,079	0,1326
4	47,270 ≤ x < 60,772	0,096	0,2286
5	60,772 ≤ x < 74,274	0,1064	0,335
6	74,274 ≤ x < 87,776	0,1034	0,4384
7	87,776 ≤ x < 101,278	0,1038	0,5422
8	101,278 ≤ x < 114,780	0,0904	0,6326
9	114,780 ≤ x < 128,282	0,076	0,7086
10	128,282 ≤ x < 141,784	0,0626	0,7712
11	141,784 ≤ x < 155,286	0,0562	0,8274
12	155,286 ≤ x < 168,788	0,0484	0,8758
13	168,788 ≤ x < 182,290	0,0306	0,9064
14	182,290 ≤ x < 195,792	0,0238	0,9302
15	195,792 ≤ x < 209,294	0,0168	0,947
16	209,294 ≤ x < 222,796	0,018	0,965
17	222,796 ≤ x < 236,298	0,0092	0,9742
18	236,298 ≤ x < 249,800	0,006	0,9802
19	249,800 ≤ x < 263,302	0,004	0,9842
20	263,302 ≤ x < 276,804	0,0038	0,988
21	276,804 ≤ x < 290,306	0,0042	0,9922
22	290,306 ≤ x < 303,808	0,002	0,9942
23	303,808 ≤ x < 317,310	0,0022	0,9964
24	317,310 ≤ x < 330,812	0,0006	0,997
25	330,812 ≤ x < 344,314	0,0008	0,9978
26	344,314 ≤ x < 357,816	0,0006	0,9984
27	357,816 ≤ x < 371,318	0,0002	0,9986
28	371,318 ≤ x < 384,820	0,0008	0,9994
29	384,820 ≤ x < 398,322	0	0,9994
30	398,322 ≤ x < 411,824	0,0002	0,9996
31	411,824 ≤ x < 425,326	0	0,9996
32	425,326 ≤ x < 438,828	0	0,9996
33	438,828 ≤ x < 452,330	0	0,9996
34	452,330 ≤ x < 465,832	0,0002	0,9998
35	465,832 ≤ x < ∞	0,0002	1

WORKING WITH EMPIRICAL DISTRIBUTIONS (II)



EXAMPLE: GENERATING A SAMPLE FROM AN EMPIRICAL DISTRIBUTION

1. Generate u uniformly distributed in [0,1]
2. Identify to which class it belongs: $u \in [F(x_j), F(x_{j+1})]$
3. Calculate

$$x = x_j + \frac{x_{j+1} - x_j}{F(x_{j+1}) - F(x_j)} \left[u - F(x_j)\right] \qquad X = \frac{\left[F(X_{i+1}) - U\right]X_i + \left[U - F(X_i)\right]X_{i+1}}{F(X_{i+1}) - F(X_i)}$$

Example: $u=0.6148 \Rightarrow u \in [0.5422, 0.6326]$

$a_{j-1}=101.278$, $a_j= 114.780$

$F(a_{j-1})= 0.5422$, $F(a_j) = 0.6326$

$$x = 101.278 + \frac{114.780 - 101.278}{0.6326 - 0.5422}\left(0.6148 - 0.5422\right) = 112.1214$$

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Inverse transformation method

Advantages:

- Intuitively easy to understand.
- Helps in variance reduction.
- Helps in generating rank order statistics.
- Helps in generating random variates from truncated distributions.

Disadvantages:

- Closed form expression for F^{-1} may not be readily available for all distributions.
- May not be the fastest and the most efficient way of generating random variates.

RANDOM VARIABLE GENERATION: Composition method

- Applicable when the desired distribution function can be expressed as a convex combination of several distribution functions.

$$F(x) = \sum_{j=1}^{\infty} p_j F_j(x),$$

where $p_j \geq 0, \sum_{j=1}^{\infty} p_j = 1$; and each F_j is a distribution function.

The general composition algorithm is:

1. Generate a positive random integer J such that:
$$\Pr\{J = j\} = p_j,\ j = 1,2,\ldots$$
2. Return X with distribution function F_J.

RANDOM VARIABLE GENERATION: Acceptance-Rejection technique

- All the previous methods were direct methods – they dealt directly with the desired distribution function.
  - This method is a bit indirect.
  - Applicable to continuous as well as discrete case.

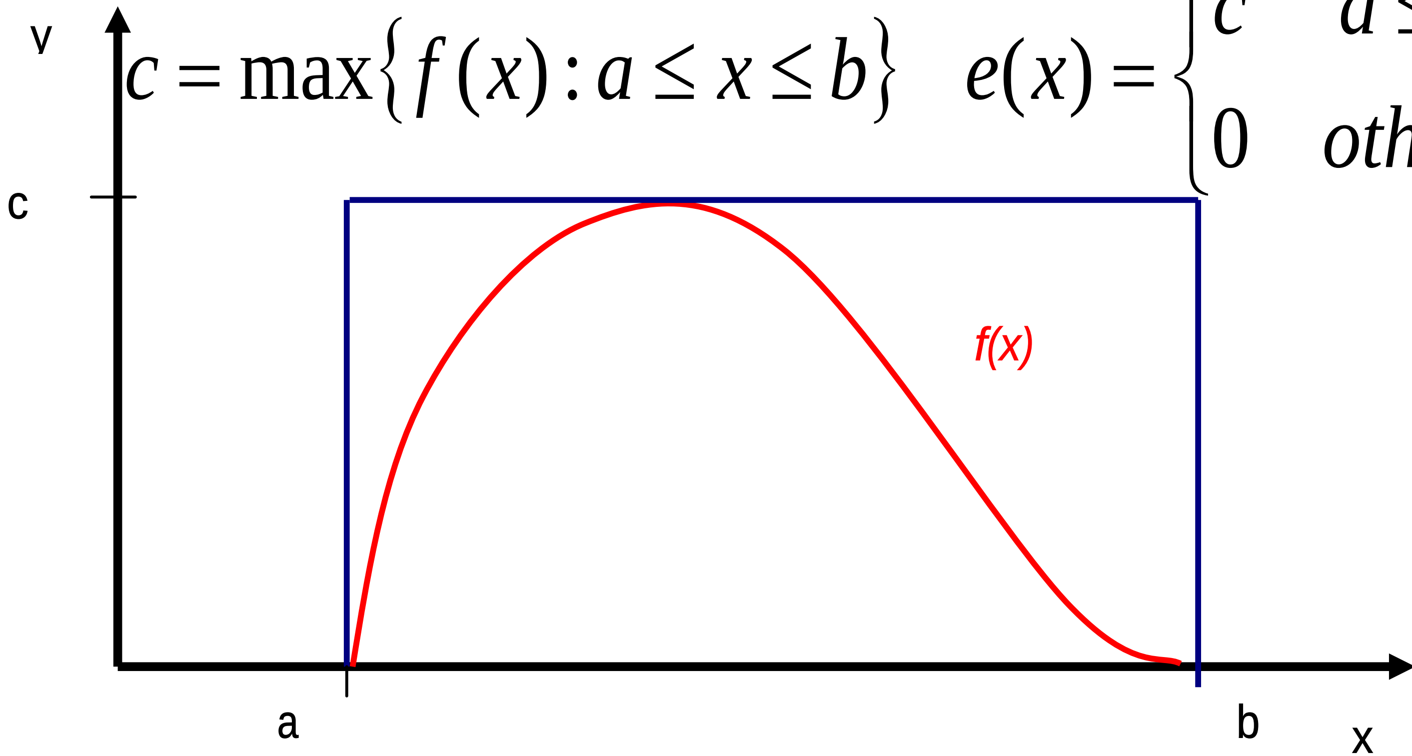
- Let f(x) be pdf of X random variable to be generated:
- We need to specify a function *e* such that $f(x) \leq e(x) \;\; \forall\, x.$ We say that *e majorizes* density *f*. In general, function *e(x)* will not be a density function, because:

$$a = \int_{-\infty}^{\infty} e(x)dx \geq \int_{-\infty}^{\infty} f(x)dx = 1.$$

- However, the function *g(x) = e(x)/a* clearly will be a density.

RANDOM VARIABLE GENERATION: Acceptance-Rejection method

$$c = \max\{f(x) : a \le x \le b\} \qquad e(x) = \begin{cases} c & a \le x \le b \\ 0 & otherwise \end{cases}$$



Algorithm:

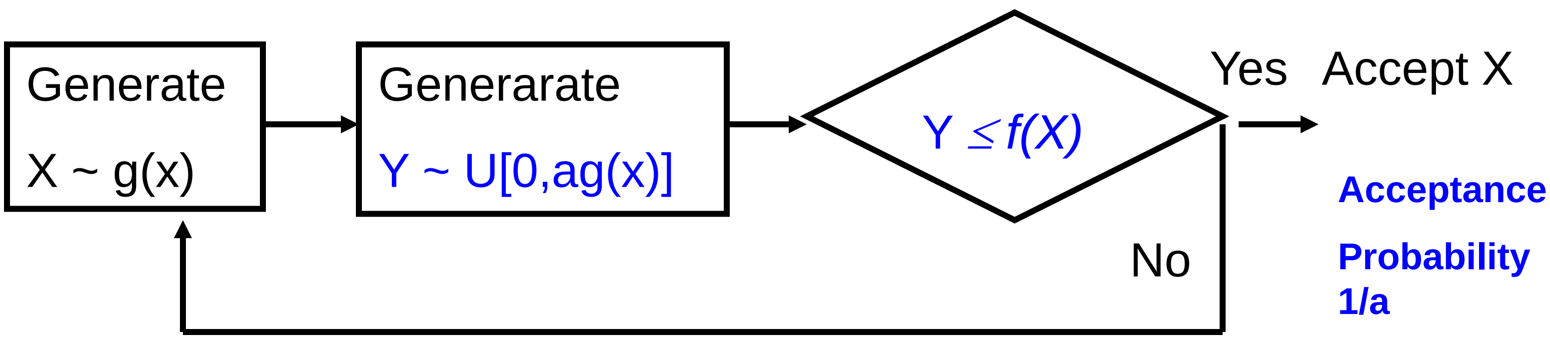
1. **Generate *X* uniform U(a,b);**
2. **Generar *Y* uniform U(0,c);**
3. **If *Y≤f(X)*, then Return *X*, otherwise to go 1.**

Inefficient if rejection occurs frequently

RANDOM VARIABLE GENERATION: Acceptance-Rejection method

The Generalized Acceptance-Rejection method:

1. Generate *X* distributed as *g(x)*; e(x)=ag(x)
2. Generate *Y* uniformly distributed in (0,*ag(X)*);
3. If $Y \leq f(X)$, then accept *X*, otherwise GOTO 1.



The Generalized Acceptance-Rejection method:

1. Generate *X* distributed as *g(x)*; e(x)=ag(x)
2. Generate *U* uniformly U(0,1);
3. If **U≤ f(X)/ag(X),** then accept *X*, otherwise GOTO 1.

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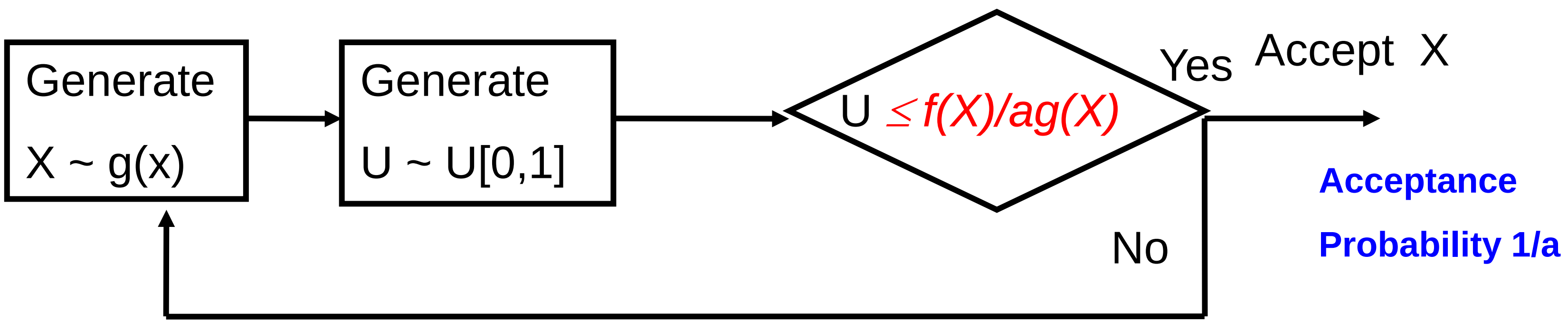

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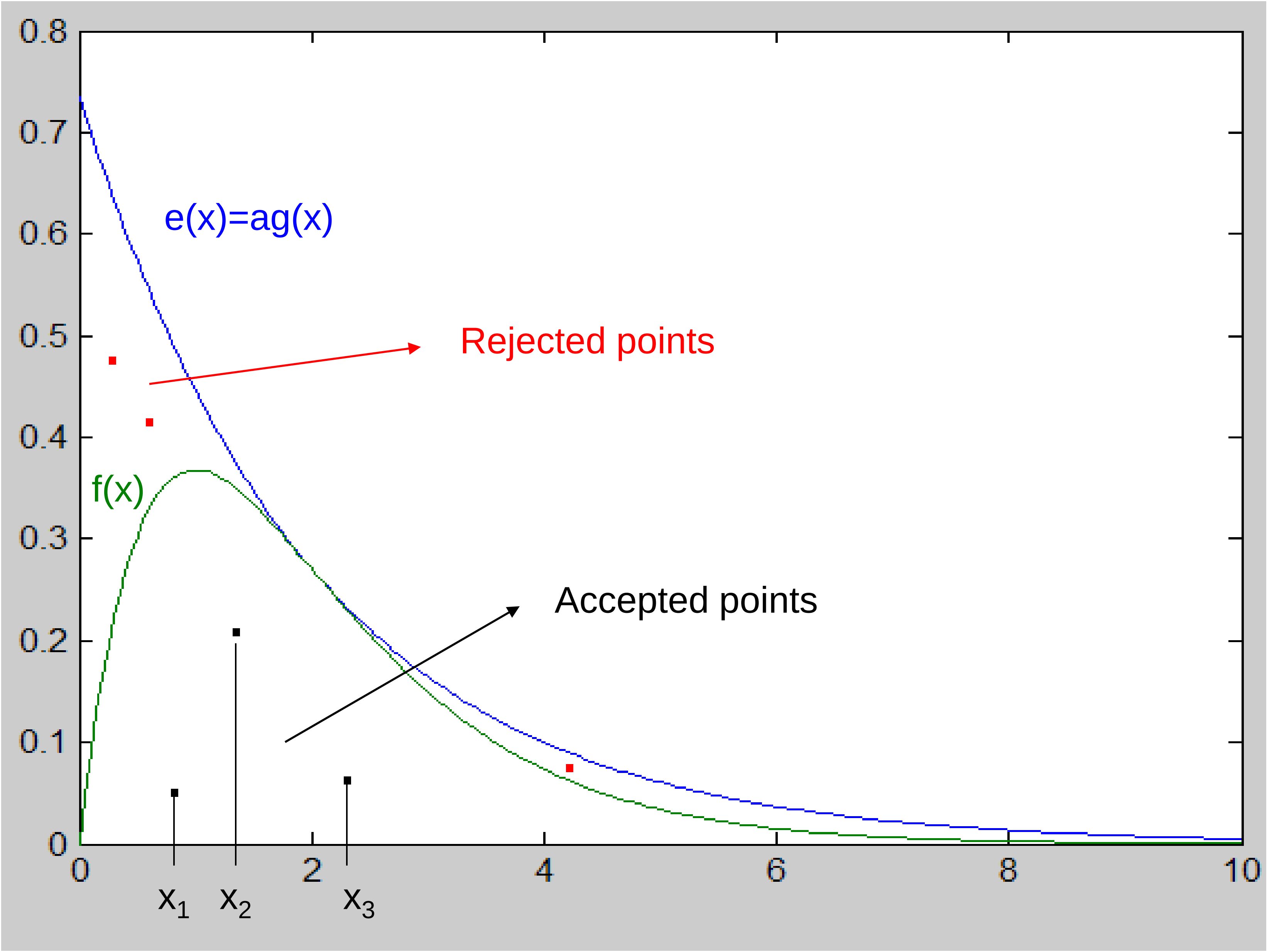
RANDOM VARIABLE GENERATION: Acceptance-Rejection method

- If X is a random variate with pdf f(x) and cdf F(x) without analytical form ($\Rightarrow$ Inverse transformation methods fail to be applied)
- There exists e(x) : e(x) $\geq$ f(x), $\forall$x
- Majorant function e(x) to be efficient
  - e(x) and f(x) are 'close' in the region area
  - e(x) = a g(x) where g(x) is pdf (probability density function) easy/cheap to generate
- It is possible to demonstrate that points

$$(X,Y) = (X, Uag(X))$$

where U is RN(0,1) are uniformly distributes in the region e(x)



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Acceptance-Rejection method

- Example: Gamma (α,β) pdf is

$$\frac{1}{\Gamma(\alpha)}\beta^{-\alpha}x^{\alpha-1}\exp\left(-\frac{x}{\beta}\right)$$

- In the standard case, β=1

$$\frac{1}{\Gamma(\alpha)}x^{\alpha-1}\exp(-x)$$

- Fishman suggested an exponential majorant

$$e(x)=\frac{a}{\alpha}\exp\left(-\frac{x}{\alpha}\right) \quad \text{with} \quad a=\frac{\alpha^{\alpha}\exp(1-\alpha)}{\Gamma(\alpha)} \quad \rightarrow e(x)\geq f(x), \forall x \geq 0$$

- For α shape parameter close to 1 then a is also close to 1, but increases as α increases

  a = 1 for α=1; a=1,83 for α=3; a=4,18 for α=15

- Suitable for moderate shape parameter α

Generation of Beta samples

Utilize the acceptance-rejection method to generate samples from a random variable X whose probability function is:

$$f(x) = 20x(1-x)^3,\ 0 < x < 1$$

(Beta Function with de parameters 2 and 4)

Since it is defined in (0,1) let's consider the rejection method with

g(x)=1, 0 < x < 1

To determine the smallest constant a such that:

$$\frac{f(x)}{g(x)} \leq a$$

We must find the maximum of : $\dfrac{f(x)}{g(x)} = 20x(1-x)^3$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = 20(1-x)^3 - 60x(1-x)^2 = 0 \quad \Rightarrow \tilde{x} = \frac{1}{4}$$

$$\frac{f(\tilde{x})}{g(\tilde{x})} = 20\frac{1}{4}\left(1-\frac{1}{4}\right)^3 = \frac{135}{64} \quad \Rightarrow a = \frac{135}{64}$$

$$\frac{f(x)}{ag(x)} = \frac{256}{27}x(1-x)^3$$

ALGORITHM:

STEP 1: Generate the uniform random numbers U_1 and U_2

STEP 2: If $U_2 \leq \dfrac{256}{27}U_1(1-U_1)^3$ **Do X=U_1**

Otherwise return to STEP 1

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Sampling from Gamma (3/2, 1)

$$f(x)=\frac{1}{\Gamma(\alpha)}\beta^{-\alpha}x^{\alpha-1}e^{-x/\beta}=\frac{1}{\Gamma\left(\frac{3}{2}\right)}x^{1/2}e^{x}=Kx^{1/2}e^{-x},x>0,K=\frac{1}{\Gamma\left(\frac{3}{2}\right)}=\frac{2}{\sqrt{\pi}}$$

The fact that the mean of the Gamma function $\Gamma(\alpha,\beta)$ equals to $\alpha\beta$ (=3/2 in this case) suggests to probe as majorant an exponential function with the same mean: $g(x)=\frac{2}{3}e^{-2x/3},x>0$

In which case: $\frac{f(x)}{g(x)}=\frac{Kx^{1/2}e^{-x}}{2/3\, e^{-2x/3}}=\frac{3K}{2}x^{1/2}e^{-x/3}$

An to calculate the constant a we have to find the maximum of $\frac{f(x)}{g(x)}$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{3K}{2}\left[\frac{1}{2}x^{-1/2}e^{-x/3}-\frac{1}{3}x^{1/2}e^{-x/3}\right]=0\Rightarrow\frac{1}{2}x^{-1/2}-\frac{1}{3}x^{1/2}=0\Rightarrow\bar{x}=\frac{3}{2}$$

And thus: $a=MAX\left[\frac{f(x)}{g(x)}\right]=\frac{f(\bar{x})}{g(\bar{x})}=\frac{3K}{2}\left(\frac{3}{2}\right)^{1/2}e^{-1/2}=\frac{3^{3/2}}{(2e\pi)^{1/2}}$ And then: $\frac{f(x)}{ag(x)}=\frac{Kx^{1/2}e^{-x}}{\frac{3K}{2}\left(\frac{3}{2}\right)^{1/2}e^{-1/2}\left(\frac{2}{3}\right)e^{-2x/3}}=\left(\frac{2e}{3}\right)^{1/2}x^{1/2}e^{-x/3}$

ALGORITHM:

STEP 1: Generate a uniform random number $U_1 \in RN(0,1)$; Do $Y=-\frac{3}{2}\ln U_1$

STEP 2: Generate a uniform random number $U_2 \in RN(0,1)$

STEP 3: If $U_2\le\left(\frac{2e}{3}\right)^{1/2}Y^{1/2}e^{-Y/3}$ **do X=Y**

Otherwise return to STEP 1

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GENERATING SAMPLES FROM A NORMAL STANDARD RANDOM VARIATE Z(0,1) (I)

Using as majoring function the exponential with mean 1: $g(x) = e^{-x}$, $0 < x < \infty$ results:

$\frac{f(x)}{g(x)} = \sqrt{\frac{2}{\pi}} e^{x - \frac{x^2}{2}}$ reaching its máximum at:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{d}{dx}\left[\sqrt{\frac{2}{\pi}} e^{x - \frac{x^2}{2}}\right] = 0 \equiv MAX\left[x - \frac{x^2}{2}\right] \Rightarrow \overline{x} = 1$$

And thus: $\frac{f(\overline{x})}{g(\overline{x})} = \sqrt{\frac{2}{\pi} e^{\frac{1}{2}}} = \sqrt{\frac{2e}{\pi}} = a \rightarrow \frac{f(x)}{ag(x)} = \exp\left\{x - \frac{x^2}{2} - \frac{1}{2}\right\} = \exp\left\{-\frac{(x-1)^2}{2}\right\}$

And therefore the algorithm to generate samples of the absolute value of the random varíate Z normal standard is:

ALGORITHM

STEP 1: Generate an uniform random number U_1; do Y=-ln U_1
STEP 2: Generate an uniform random number U_2

STEP 3: If $U_2 \leq \exp\left\{-\frac{(Y-1)^2}{2}\right\}$ **do X=Y**

Otherwise return to STEP 1

The standard normal Z can be obtained making that it be X *or* –X *with the same probability*

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GENERATING SAMPLES FROM A NORMAL STANDARD RANDOM VARIATE Z(0,1) (II)

ALGORITHM:

STEP 1: Generate an uniform random number U_1; do Y_1=-ln U_1
STEP 2: Generate an uniform random number U_2; do Y_2=-ln U_2
STEP 3: If $Y_2 - \frac{(Y_1 - 1)^2}{2} > 0$ **Do Y=** $Y_2 - \frac{(Y_1 - 1)^2}{2}$ **; GO TO STEP 4**
Otherwise return to STEP 1
STEP 4: Generate an uniform random number U and do:

$$Z = \begin{cases} Y_1 & \text{Si } U \leq \frac{1}{2} \\ -Y_1 & \text{Si } U > \frac{1}{2} \end{cases}$$

To generate a normal random variable X~N(μ,σ)do the transform:

$$X = \mu + \sigma Z$$

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GAMMA DISTRIBUTION ($\alpha > 1$, Cheng)

Probability function

$$f(x) = \begin{cases} \dfrac{\beta^{-\alpha}(x)^{\alpha-1}}{\Gamma(\alpha)} \exp\left(-\dfrac{x}{\beta}\right) & x > 0 \\ 0 & \text{Otherwise} \end{cases}$$

$$\text{where } \Gamma(\alpha) \text{ is the Gamma function} : \Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u} du$$

$$\text{Do} : a = (2\alpha - 1)^{-1/2},\ b = \alpha - \ln 4, c = \alpha + a^{-1}, d = 1 + \ln 4.5$$

While (True){

Do $U_1 = RN(0,1)$ $U_2 = RN(0,1)$

Do $V = a \ln\left[U_1 / (1 - U_1)\right]$, $Y = \alpha e^V$, $Z = U_1 U_2$, $W = b + cV - Y$

If $(W + d - 4.5Z \geq 0)$ {

Return $X = \beta Y$

Otherwise

If $(W \geq \ln Z)$ Return $X = \beta Y$

}

}

GAMMA DISTRIBUTION: simple generator $1<\alpha<5$ (Fishman)

$$
\begin{aligned}
&\text{While (True)} \{ \\
&\quad \text{Do } U_1 = RN(0,1),\ U_2 = RN(0,1) \\
&\quad\quad V_1 = -\ln U_1,\quad V_2 = -\ln U_2 \\
&\quad \text{If } (V_2 > (\alpha - 1)(V_1 - \ln V_1 - 1))\{ \\
&\quad\quad \text{Return } X = \beta V_1 \\
&\quad\quad \} \\
&\quad \}
\end{aligned}
$$

GENERATING POISSON SAMPLES BY THE REJECTION METHOD

- **A RANDOM POISSON VARIABLE N, DISCRETE WITH MEAN $\lambda>0$ HAS A PROBABILITY FUNCTION**

$$p(n) = P(N = n) = \frac{e^{-\lambda}\lambda^n}{n!}, n = 0,1,2,.....$$

- **N CAN BE INTERPRETED AS THE NUMBER OF POISSON ARRIVALS IN A UNIT TIME INTERVAL AND THEREFORE THE INTERARRIVAL TIMES A_1, A_2, ... WILL BE EXPONENTIALLY DISTRIBUTED WITH MEAN $1/\lambda$, AND THUS: N = n**
- **IF AND ONLY IF: $A_1+A_2+......+A_n \leq 1 < A_1+A_2+......+A_n+A_{n+1}$.**
- **AND TAKING INTO ACCOUNT THAT $A_i = -(1/\lambda)\ln(u_i)$ RESULTS**

$$\sum_{i=1}^{n} \frac{-1}{\lambda} \ln u_i \leq 1 < \sum_{i=1}^{n+1} \frac{-1}{\lambda} \ln u_i$$

$$\ln \prod_{1}^{n} u_i = \sum_{i=1}^{n} \ln u_i \geq -\lambda > \sum_{i=1}^{n+1} \ln u_i = \ln \prod_{1}^{n+1} u_i$$

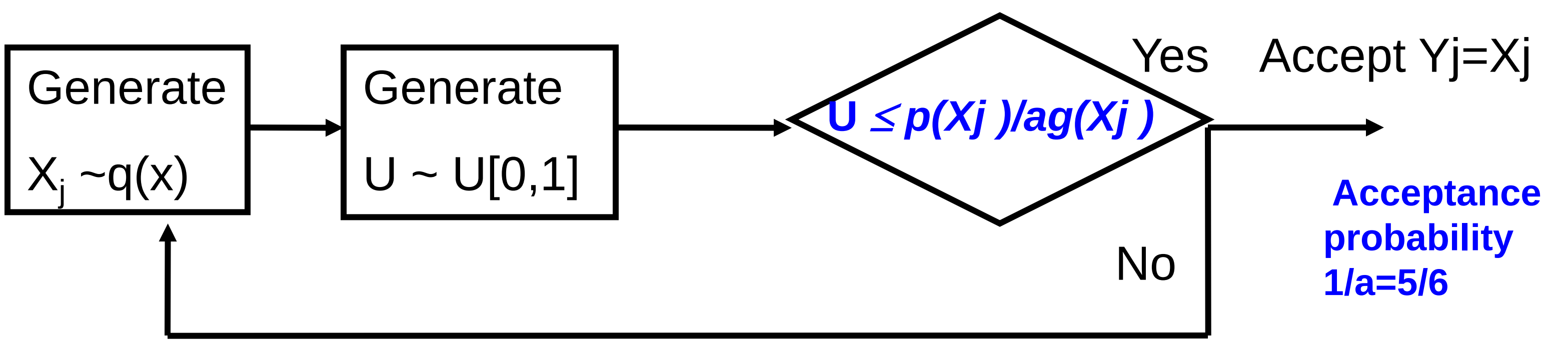
$$\prod_{1}^{n} u_i \geq e^{-\lambda} > \prod_{1}^{n+1} u_i$$

- **ALGORITHM:**
  - **STEP 1: DO n = 0, P = 1**
  - **STEP 2: GENERATE AN UNIFORM RANDOM NUMBER u_{n+1} AND REPLACE P BY u_{n+1} P**
  - **STEP 3: IF P < $e^{-\lambda}$ THEN ACCEPT N = n. OTHERWISE REJECT THE CURRENT n, INCREASE n BY ONE UNIT AND REPEAT FROM STEP 2.**

ACCEPTANCE REJECTION METHOD FOR DISCRERE RANDOM VARIABLE (DRV)

- Let Y be a DRV with k values and probability function $p(y_j)=p_j$.
- Let X be a DRV with k values and probability function $q(x_j)=q_j$ easy to generate.
- Example: Y_j and X_j values 1 to 10, con funciones de probabilidad: q(x) = {0.1,…,0.1} and p(y)={0.11,0.12,0.09,0.08,0.12,0.1,0.09,0.09,0.1,0.1}
- Let a be such that $p(Xj) / q(Xj) \leq a$, a=1.2

$$U \leq p(Xj)/ag(Xj) = p(Xj)/0.12$$



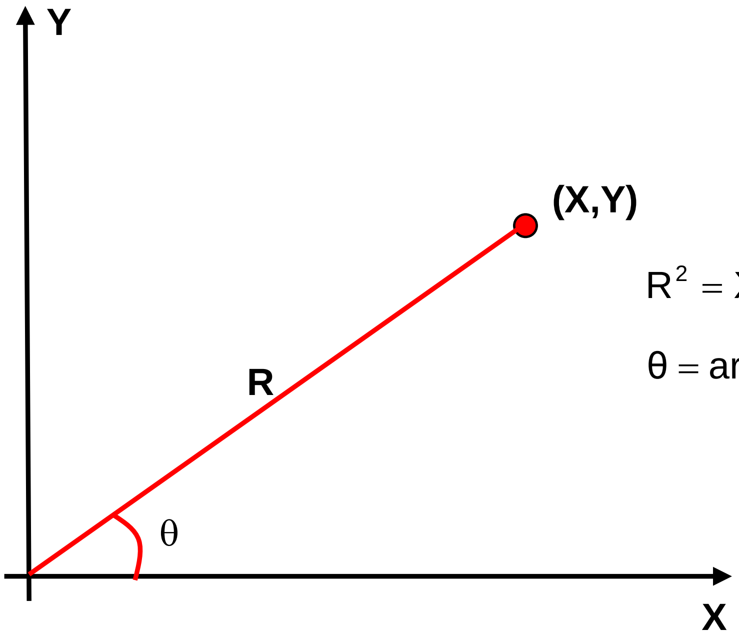
OTHER METHODS

- **GENERATING AN ERLANG e(k,μ) DISTRIBUTION AS SUM OF k EXPONENTIAL, INDEPENDENT RANDOM VARIABLES x_i, IDENTICALLY DISTRIBUTED WITH MEAN 1/kμ**

$$X = \sum_{i=1}^{k} X_i = -\frac{1}{k\mu} \ln\left(\prod_{i=1}^{k} u_i \right)$$

METHOD OF BOX AND MULLER TO GENERATE SAMPLES OF THE STANDARD NORMAL RANDOM VARIABLE (I)

Let X and Y two normal standard independent random variables. Let's denote byr R and θ the polar coordinates of point (X,Y)



$$R^2 = X^2 + Y^2 \qquad (X = R\cos\theta)$$

$$\theta = \text{arc tg}\,\frac{Y}{X} \qquad (Y = R\text{sen}\theta)$$

Given that X and Y are independent their joint probability function will be the product of the individual probability functions:

$$f(x,y) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} = \frac{1}{2\pi} e^{-(x^2+y^2)/2}$$

To determine the joint probability function of R^2 and θ, f(R,θ) let's do the variable change:

$$d = x^2 + y^2 \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad \text{and then} \quad f(d,\theta) = |J|^{-1} f(x,y)$$

Where J is the Jacobian of the transformation, J=2

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METHOD OF BOX AND MULLER TO GENERATE SAMPLES OF THE STANDARD NORMAL RANDOM VARIABLE (II)

$$J = \begin{vmatrix} \frac{\partial d}{\partial x} & \frac{\partial d}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} = \frac{2x^2}{x^2+y^2} + \frac{2y^2}{x^2+y^2} = 2$$

And thus $f(d,\theta) = \frac{1}{2}\frac{1}{2\pi} e^{-\frac{d}{2}}, 0 < d < \infty; 0 < \theta < 2\pi$ which is equal to the product of a random uniform function in $(0,2\pi)$ and an exponential of mean 2, $\left(\frac{1}{2} e^{-\frac{d}{2}}\right)$, what implies that R^2 y θ are independent with R^2 exponential of mean 2 and θ uniformly distributed in $(0,2\pi)$.

- **A pair X, Y of normal standard independent random variables can be generated by generating R^2 θ, polar coordinates of point (X,Y) and transforming them to cartesian coordinates.**

METHOD OF BOX AND MULLER TO GENERATE SAMPLES OF THE STANDARD NORMAL RANDOM VARIABLE (III)

BOX AND MULLER ALGORITHM

STEP 1: Generate the independent random numbers U_1 y U_2
PASO 2: Generate R^2, exponential with mean 2: $R^2 = -2 \ln U_1$
Generate θ uniform in $(0,2\pi)$: $\theta = 2\pi U_2$
PASO 3: Change to cartesian coordinates:

$$X = R\cos\theta = \sqrt{-2\ln U_1}\cos\left(2\pi\pi_2\right)$$

$$Y = R\operatorname{sen}\theta = \sqrt{-2\ln U_1}\operatorname{sen}\left(2\pi\pi_2\right)$$

NORMAL DISTRIBUTION: Acceptance-Rejection method (Marsaglia)

- Let d be a positive scalar , **generate X standard normal conditional to X≥d**
- The probability density function f(x) for X can be expressed as

  $f(x)=c.\exp(-x^2/2)$
- Where c can be computed.
- A suitable g(x) might be $g(x)=x.\exp(-[x^2-d^2]/2)$ (squared root of an exponencial variate with mean 2 and shifted d^2).
  - Let Y be an exponencial variate with mean 2 then pdf h(y)= exp(-y/2)/2 and let us define X:

$$X=\sqrt{Y+d^2} \rightarrow g(x)=|J|^{-1}h(y)$$

$$|J|^{-1}=\left(\frac{dx}{dy}\right)^{-1}=\left(\frac{d\sqrt{Y+d^2}}{dy}\right)^{-1}=\left(\frac{1}{2\sqrt{Y+d^2}}\right)^{-1}\xrightarrow{Y=X^2-d^2}\left(\frac{1}{2\sqrt{X^2}}\right)^{-1}=2X$$

  - And thus, g(x):

$$g(x)=|J|^{-1}h(y)=(2x)\exp(-y/2)/2\xrightarrow{\text{in X}}(2x)\exp(-\tfrac{X^2-d^2}{2})/2=x\exp(-\tfrac{X^2-d^2}{2})$$

NORMAL DISTRIBUTION: Acceptance-Rejection method (Marsaglia)

- (Cont.) Let d be a positive scalar , generate X standard normal conditional to X≥d

- Compute the constant a in the definition of the majorant e(x)=a g(x):

$$\frac{ag(x)}{f(x)} \geq 1; x \geq d \text{ so } \frac{ax.\exp\left(-\left[x^2-d^2\right]/2\right)}{c.\exp\left(-x^2/2\right)} \geq 1 \Rightarrow x \geq \frac{c}{a}\exp\left(-d^2/2\right)$$

- Choose **a** to satisfy equality : maximun for **x=d**

$$\frac{c}{a}\exp\left(-d^2/2\right) = d$$

- Generate X distributed with pdf g(x):

$$X = \sqrt{\left(d^2 - 2\ln\left(1-U_1\right)\right)} \quad \left(\equiv \sqrt{\left(d^2 - 2\ln\left(U_1\right)\right)}\right)$$

- Accept X if

$$U_2 ag(X) \leq f(X) \Rightarrow U_2 aX\exp\left(-\left[X^2-d^2\right]/2\right) \leq c.\exp\left(-X^2/2\right) \Leftrightarrow U_2 X \leq d$$

NORMAL DISTRIBUTION: Acceptance-Rejection method (Marsaglia)

1. Generate U_1 y U_2 distributed U(0,1)
2. Let $X = \sqrt{\left(d^2 - 2\ln(U_1)\right)}$
3. If $U_2 X \leq d$, then accept X; otherwise GOTO 1